Exercises on Lie groups

Spring term 2018, Sheet 6

Hand in before 10 o'clock on 13th April 2018
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Exercise 1.
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

(i) Show that G is abelian, then $\mathfrak{g}$ is abelian.

(ii) Assuming that G is connected, show that if $\mathfrak{g}$ is abelian, then G is abelian.

(iii) Provide an example of a non-abelian Lie groups whose Lie algebra is abelian.

Exercise 2.
We show that classification of Lie algebras in practice asks some work.

(i) Classify all 2-dimensional Lie algebras over $\mathbb{R}$ up to isomorphism.

(ii) Determine all non-abelian 2-dimensional Lie subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ up to conjugacy by elements of $\mathrm{GL}(3, \mathbb{R})$ and provide a basis for a representant in class.
Hint: use the Jordan normal form.

Exercise 3.
In this exercise we show that the exponential map is not necessarily surjective, even on the connected component of the identity.
Fix the matrix

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \, .$$

(i) Show that A is connected to the identity of $\mathrm{GL}(2, \mathbb{R})$.

(ii) Show that A is not a square in $\mathrm{GL}(2, \mathbb{R})$.

(iii) Show that every element in the image of the exponential map of a Lie group is a square.

(iv) Conclude that A is not in the image of the exponential map of $\mathrm{GL}(2, \mathbb{R})$.