Exercises on Lie groups

Spring term 2018, Sheet 11

Hand in before 10 o'clock on 11th May 2018 | Sven Raum
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Exercise 1

Let G be a connected Lie group. Show that G is solvable if and only if $\mathrm{Lie}(G)$ is solvable.

Exercise 2

In this exercise we identify the topology on the automorphism group of a connected Lie group. Let G be a connected Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and universal cover $\tilde{G}$. We saw that there is an isomorphism of abstract groups $\mathrm{Aut}(\tilde{G}) \cong \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$.

(i) Show that $\mathrm{Aut}(\mathfrak{g})$ is closed in $\mathrm{GL}(\mathfrak{g})$

Since closed subgroups of Lie groups have a natural structure of a Lie group, we find that $\mathrm{Aut}(\tilde{G})$ is a Lie group. We want to obtain a Lie group structure on $\mathrm{Aut}(G)$. Denote by $Z \leq \tilde{G}$ the central subgroup satisfying $\tilde{G}/Z = G$. Then there is a natural isomorphism

$$\mathrm{Aut}(G) \cong \{\alpha \in \mathrm{Aut}(\tilde{G}) \mid \alpha(Z) = Z\}\,.$$

We want to show that this is a closed subgroup, so that also $\mathrm{Aut}(G)$ becomes a Lie group. For a topological group H the space $\mathrm{C}(H,H)$ of continuous maps from H to itself is endowed with the topology of uniform convergence on compact subsets whose basis is

$$N_{C,U,f_0} = \{f \in \mathrm{C}(H,H) \mid \forall g \in : f(g)f_0^{-1}(g) \in U\}$$

for $C \subset H$ compact, $U \subset H$ open and $f_0 \in \mathrm{C}(H,H)$.

(ii) Let H be a connected simply connected Lie group. Show that $\mathrm{Aut}(H)$ has the subspace topology inherited from $\mathrm{Aut}(H) \subset \mathrm{C}(H,H)$.

(iii) Conclude that $\mathrm{Aut}(G) \subset \mathrm{Aut}(\tilde{G})$ is closed

We can now endow $\mathrm{Aut}(G)$ with the structure of a Lie group. It remains to identify its topology in the same way as we just did for conneced simply connected Lie groups.

(iv) Let H be a connected Lie group. Show that $\mathrm{Aut}(H)$ has the subspace topology inherited from $\mathrm{Aut}(H) \subset \mathrm{C}(H,H)$.

Exercise 3

In this exercise we investigate different aspects of the semi-direct product construction for Lie groups. Let H, N be Lie groups and $\alpha : H \to \mathrm{Aut}(N)$ a Lie group homomorphism.

(i) Show that the group theoretical semi-direct product $N \rtimes_\alpha H$ becomes a Lie group when equipped with the differential structure of the manifold $N \times H$.

(ii) Show the universal property of the semi-direct product: for every pair of Lie group homomrphisms $\varphi_N : N \to G$ and $\varphi_H : H \to G$ such that $\varphi_H(h)\varphi_N(n)\varphi_H(h)^{-1} = \varphi_N(\alpha_h(n))$ for all $n \in N$, $h \in H$ there is a unique extension to a Lie group homomorphism of $N \rtimes_\alpha H$

Exercise 4.

In this exercise we investigate the semi-direct product decomposition of a Lie group. Given a Lie group G with a closed normal subgroup $N \trianglelefteq G$ and a closed subgroup $H \leq G$ such that $N \cap H = \{e\}$ and $NH = G$, we say that $G = N \rtimes H$ is a semi-direct product.

(i) Let G be a Lie group. Show that the group of inner automorphism $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ is a Lie subgroup.

(ii) Let $N \trianglelefteq G$ be a closed normal subgroup of a Lie group. Show that the map $G \to \mathrm{Aut}(N)$ induced by conjugation is a Lie group homomorphism.

(iii) Let $N \leq G$ be a closed normal Lie subgroups and $H \leq G$ be some Lie subgroup. Show that $G = N \rtimes H$ if and only if there is an isomorphism $N \rtimes_{Ad} H \to G$ extending the inclusions of N and H.