Exercises on Homology and Cohomology

Spring term 2018, Sheet 10

Hand in before 10 o'clock on 7th May 2018
Mailbox of Sven Raum in MA B2 475

Sven Raum
Haoqing Wu

Exercise 1 (easy)
Let G be a group and R a ring. Show that

$$\mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}G, R) \cong \mathrm{Hom}_{\mathrm{Grps}}(G, R^{\times}),$$

where $R^{\times}$ denotes the group of units of R.

Exercise 2 (medium)
Let F be a free R-module and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules. Show that

$$0 \to \operatorname*{Hom}_R(F, M_1) \to \operatorname*{Hom}_R(F, M_2) \to \operatorname*{Hom}_R(F, M_3) \to 0$$

is a short exact sequence of abelian groups.

Exercise 3 (difficult)
In this exercise we find a natural CW-complex on which the free group $\mathbb{F}_2$ acts freely and thus calculate its cohomology. This is the so-called Cayley tree of $\mathbb{F}_2$. Let $S = \{a, b, \}$ be a set of generators of $\mathbb{F}_2$. We consider the directed graph T whose vertex set is $\mathrm{V}(T) = \mathbb{F}_2$ and whose edge set is $\mathrm{E}(T) = \{(g, gs) \mid g \in \mathrm{V}(T), s \in S\}$.

(i) Show that there is a well-defined CW-complex with 0-cells equals to $\mathrm{V}(T)$ and 1-cells equal to $\mathrm{E}(T)$, with gluing maps $\varphi_{(g,gs)} : \mathrm{S}^0 = \{-1, 1\} \to \mathrm{V}(T)$ given by $\varphi_{(g,gs)}(-1) = g$ and $\varphi_{(g,gs)}(1) = gs$. This CW-complex is called the geometric realisation of T and it is denoted $|T|$.

(ii) Show that $|T|$ is contractible.

(iii) Show that the actions $\mathbb{F}_2 \curvearrowright \mathrm{V}(T), \mathrm{E}(T)$ by left multiplication turn $|T|$ into a contractible free $\mathbb{F}_2$-CW-complex.

(iv) Show that $|T|/\mathbb{F}_2$ is a bouquet of two circles $\mathrm{S}^1 \vee \mathrm{S}^1$.

(v) Calculate the cohomology of $\mathrm{S}^1 \vee \mathrm{S}^1$. This is the cohomology of $\mathbb{F}_2$.