Character rigidity for lattices and commensurators II
after Creutz-Peterson

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1 Introduction

The aim of talks C3 and C4 is to give a proof of the following theorem.

Theorem 1 ([CP13]). *Let G be a simple non-compact Lie group with property (T) and trivial centre and let H be a product of simple Howe-Moore groups. Let Λ be an irreducible lattice in $G \times H$.*

Then every extremal character $\tau : \Lambda \to \mathbb{C}$ is either almost periodic or $\tau = \delta_e$ is the left-regular character.

In lecture C3 we saw that it suffice to show the following von Neumann algebraic reformulation of Theorem 1.

Theorem 2 (Theorem B of [CP13]). *Let G be a simple non-compact Lie group with property (T) and trivial centre. Assume that Λ is a countable dense subgroup which contains and commensurates a lattice Γ of G and such that $\Lambda//\Gamma$ is a product of simple groups with the Howe-Moore property.*

If $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation of Λ such that $\pi(\Lambda)'' = M$, then

- *either M is finite dimensional or*
- *π extends to an isomorphism $\mathrm{L}(\Lambda) \to M$.*

Actually, in the situation of Theorem 2, we already showed that M is finite dimensional if we assume that $\pi(\Gamma)''$ is amenable. It hence remains to prove the following statement.

Theorem 3. *Let G be a simple non-compact Lie group with property (T) and trivial centre. Assume that Λ is a countable dense subgroup which contains and commensurates a lattice Γ of G.*

Assume that $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation of Λ such that $\pi(\Lambda)'' = M$, but π does not extend to an isomorphism $\mathrm{L}(\Lambda) \to M$. Then $\pi(\Gamma)''$ is amenable.

Note that in this lecture we don't make use of the assumption that $\Lambda//\Gamma$ is a product of simple groups with the Howe-Moore property.

[1] last modified April 9, 2014

2 Preparations

Non-degenerate representations. We say that a representation $\pi : \Upsilon \to \mathcal{U}(M)$ of a discrete group into a von Neumann algebra is *non-degenerate* if $M = \pi(\Upsilon)''$.

Poisson boundary. Let us recall that the Poisson boundary of G is the quotient G/P of G by a minimal parabolic subgroup. We denote by $G \overset{\sigma}{\curvearrowright} (B, \eta)$ the non-singular action on G/P with respect to the measure class induce by the Haar measure of G. The only properties of $G \curvearrowright B$ that we will exploit during this lecture are amenability and contractivity. Let us start by explaining contractive actions.

Contractive actions. Let $G \overset{\sigma}{\curvearrowright} (B, \eta)$ be a non-singular action of a discrete group. Then $G \curvearrowright B$ is called contractive if for all measurable $A \subset B$ we have $\sup_{g \in G} \mu(gA) \in \{0, 1\}$. Contractive actions have remarkable rigidity properties, which are explained in Section 4.

Amenability of the noncommutative Poisson boundary. We will not explain amenability of $G \overset{\sigma}{\curvearrowright} B$, but rather state one of its consequences. Given a non-degenerate representation $\pi : \Gamma \to N$ into a tracial von Neumann algebra, we introduce the algebra of Γ-equivariant measurable functions from B into $\mathcal{B}(\mathrm{L}^2(N))$.

$$\mathcal{B}_N = \{\sigma_g \otimes J\pi(g)J \mid g \in \Gamma\}' \cap \mathrm{L}^\infty(B) \overline{\otimes} \mathcal{B}(\mathrm{L}^2(N)) \, .$$

Here $\sigma_g \in \mathcal{U}(\mathrm{L}^2(B, \eta))$ denote the unitary operator defined by the action of g on (G, η). Moreover, J is defined by the formula $Jx^* J\hat{a} = \widehat{ax}$ for all $a, x \in N$. This von Neumann algebra is, for reasons that we cannot explain during this lecture, called the *noncommutative Poisson boundary* of π. As a consequence of the amenability of $G \curvearrowright B$, it follows that $\mathcal{B}_N$ is an amenable von Neumann algebra as [Zim77, Theorem 5.1] shows.

G-algebras. Let $\Lambda \leq G$ be a countable dense subgroup of a locally compact group G. Let $\Lambda \curvearrowright M$ be an action on a tracial von Neumann algebra. The G-algebra of M is the algebra of all elements $x \in M$ such that whenever $(\lambda_n)_n$ is a sequence in Λ that converges to e in G, then $\lambda_n x \to x$ strongly. If $\pi : \Lambda \to \mathcal{U}(M)$ is a representation of Λ in M, then the G-algebra of π is the G-algebra for the action $x \mapsto \mathrm{Ad}(\pi(\lambda))(x)$. The G-algebra of an action $\Lambda \curvearrowright M$ is the largest von Neumann subalgebra of M on which we can extend the action of Λ to a continuous action of G.

Notation. If not stated differently, we use the notion of the introduction and the preparatory section throughout these notes.

3 Proof of the main theorem

In this section we give the proof of our main Theorem 3. We start by stating three results, which are the main ingredients of its proof. Their proofs in turn are postponed to later sections. The statements of Proposition 4 and Theorems 5 and 6 are chosen in such a way that the principal ingredients of the proof of the main theorem become clear.

Proposition 4 (Proposition 5.1 in [CP13]). *Let G be a second countable locally compact group and with a lattice $\Gamma \leq G$ and a countable dense subgroup $\Lambda \leq G$ that contains and commensurates Γ. Let $\pi : \Lambda \to \mathcal{U}(M)$ be a non-degenerate finite factor representation and put $N = \pi(\Gamma)''$. Then*

$$\begin{aligned} & \{\sigma_\gamma \otimes (J\pi(\gamma)J) \mid \gamma \in \Gamma\}' \cap \mathrm{L}^\infty(B) \overline{\otimes} \mathcal{B}(\mathrm{L}^2(N, \tau)) \\ = \; & \{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}' \cap \mathrm{L}^\infty(B) \overline{\otimes} \mathcal{B}(\mathrm{L}^2(N, \tau)) \end{aligned}$$

The proof of Proposition 4 relies on a rigidity phenomenon related to contractive actions. In Section 4, we give a brief outline of the relevant results. However, for a full proof we refer to [CP13, Sections 3 and 5].

Theorem 5. *Let G be a simple non-compact Lie group with trivial centre. Let $\Lambda \leq G$ be a countable dense subgroup. Assume that $\pi : \Lambda \to \mathcal{U}(M)$ is a non-degenerate finite factor representation of Λ. Then the G-algebra of π equals $\mathbb{C}1$.*

This theorem can be considered as a strengthening of Theorem 7 of talk C3, saying that a simple group with the Howe-Moore property does not have any non-trivial representation into a finite von Neumann algebra. Indeed, the Howe-Moore property is the main ingredient of the proof of Theorem 5, which is presented in Section 7.

Theorem 6 (Theorem 4.4 in [CP13])**.** *Let G be a simple locally compact group and $\Lambda \leq G$ a countable dense subgroup. Let $\pi : \Lambda \to \mathcal{U}(M)$ be a non-degenerate finite factor representation and assume that the G-algebra of π is $\mathbb{C}1$. If π does not extend to an isomorphism $\mathrm{L}(\Lambda) \to M$ then for any von Neumann subalgebra $N \subset M$ we have*

$$\{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \,|\, \lambda \in \Lambda\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N)) = 1 \otimes N\,.$$

With these means at hand, let us prove our main Theorem 3.

Proof of Theorem 3. Assume that $\pi : \Lambda \to \mathcal{U}(M)$ is a non-degenerate finite factor representation that does not extend to an isomorphism $\mathrm{L}(\Lambda) \to M$. We noted in Section 2 that the von Neumann algebra

$$\mathcal{B}_N = \{\sigma_g \otimes J\pi(g)J \,|\, g \in \Gamma\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N))$$

is amenable. Since $G \curvearrowright B$ is contractive, Proposition 4 implies that

$$\mathcal{B}_N = \{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \,|\, \lambda \in \Lambda\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N,\tau))\,.$$

By Theorem 5, the G-algebra of M is trivial. Hence we may apply Theorem 6, so as to conclude that $\mathcal{B}_N = 1 \otimes N$. This shows that N is amenable. □

4 Rigidity results for contractive actions

In this section we only briefly state how deduce Proposition 4 from contractivity of the Poisson boundary. We don't state the results of [CP13, Section 3] in their full generality.

Let us start by considering classical contractive actions. They posses an interesting approximation feature explained in the following lemma. Its proof is elementary and can be deduced right away from the definition of contractive actions.

Lemma 7. *Let $\Gamma \curvearrowright (B,\eta)$ be a contractive non-singular action of a discrete group. Take $f \in \mathrm{L}^\infty(X)_+$ and $\tilde{f} \in \mathrm{L}^\infty(X)$. Then there is a sequence of elements $(g_n)_n$ in Γ such that $g_n f \to \|f\|$ strongly and $g_n \tilde{f} \to c \in \mathbb{C}$.*

Recall the noncommutative Poisson boundary associated with a non-degenerate representation $\pi : \Gamma \to N$ into a tracial von Neumann algebra.

$$\mathcal{B}_N = \{\sigma_g \otimes J\pi(g)J \,|\, g \in \Gamma\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N))\,.$$

Lemma 7 has the following noncommutative analogue.

Lemma 8. *Let Γ be a countable discrete discrete group and let $\Gamma \overset{\sigma}{\curvearrowright} (B,\eta)$ be a contractive action and $\pi : \Gamma \to \mathcal{U}(N)$ a non-degenerate representation of Γ into a finite von Neumann algebra. Let*

$$\mathcal{B}_N = \{\sigma_g \otimes J\pi(g)J \,|\, g \in \Gamma\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N))\,.$$

Take $x \in \mathcal{B}_N$, $\|x\| \le 1$, $f \in \mathrm{L}^\infty(B)_+$ and $\varepsilon > 0$. Then there is a sequence $(g_n)_n$ in Γ, a sequence of projections $(p_n)_n$ in N satisfying $\tau(p_n) > 1 - \varepsilon$ for all $n \in \mathbb{N}$ and a sequence of elements $(y_n)_n$ in N such that $(y_n)_n$ is uniformly bounded, $\sigma_{g_n}(f) \to \|f\|$ strongly and $\pi(g_n)(p_n x - y_n)\pi(g_n^{-1}) \to 0$ strongly.

The analogy between Lemma 7 and Lemma 8 becomes clear if we consider N as a replacement for the scalars $\mathbb{C}1$. The element x in Lemma 8 corresponding to $\tilde{f}$ in Lemma 7 can be approximated by "scalars" in a suitable sense.

Using the approximation results of the previous lemma, one can deduce a rigidity result for N-bimodular maps.

Theorem 9. *If $1 \otimes N \subset P \subset \mathcal{B}_N$ is an intermediate von Neumann algebra and $\Phi : P \to \mathcal{B}_N$ is a strongly continuous unital N-bimodular map, then $\Phi = \mathrm{id}$.*

Idea of the proof. One notes that Φ is N-bimodular and hence $\Phi|_N = \mathrm{id}_N$. By Lemma 8, it is possible to approximate arbitrary elements of P by elements of N under large projections. This allows to conclude that $\Phi = \mathrm{id}_P$. □

From the last theorem, or rather a slight generalisation of it, we can deduce Proposition 4.

Propositon 4 (Recall)**.** *Let G be a second countable locally compact group, $\Gamma \le G$ a lattice and $\Lambda \le G$ a countable dense subgroup that contains and commensurates Γ. Let $G \curvearrowright (B,\eta)$ be a contractive non-singular action. Let $\pi : \Lambda \to \mathcal{U}(M)$ be a non-degenerate finite factor representation and put $N = \pi(\Gamma)''$. Then*

$$\mathcal{B}_N = \{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \,|\, \lambda \in \Lambda\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N,\tau))\,.$$

Idea of the proof. We consider N-bimodular maps looking like

$$\Psi : \mathcal{B}_N \to \mathcal{B}_N : x \mapsto \mathrm{Ad}(\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J))(x)\,.$$

Applying Theorem 9 shows that $\Psi = \mathrm{id}$ and hence $\mathcal{B}_N \subset \{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \,|\, \lambda \in \Lambda\}' \cap \mathrm{L}^\infty(B)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N,\tau))$. In the proof one meets two technical difficulties. First, $\mathrm{E}_N(\pi(\lambda))$ doesn't need to be a unitary. We can remedy this default by using a polar decomposition. Second, the image of Ψ does not necessarily lie in $\mathcal{B}_N$. Here commensurability of $\Gamma \le \Lambda$ comes into play and one has to prove a generalisation of Theorem 9 replacing $\mathcal{B}_N$ by an analogue for finite index subgroups of Γ. □

5 Triviality of the noncommutative Poisson boundary for non-regular representations

This section is devoted to the proof of Theorem 6. It is the only part of the proof of our main Theorem 3 where we use the fact that the representation $\pi : \Lambda \to M$ does not extend to an isomorphism $\mathrm{L}(\Lambda) \to M$. Moreover, it demonstrates some useful von Neumann algebra techniques.

Theorem 6 (Recall)**.** *Let G be a simple locally compact group and $\Lambda \leq G$ be a countable dense subgroup. Suppose that $G \curvearrowright Y$ is an ergodic non-singular action and $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation such that $M = \pi(N)''$. Assume that the G-algebra with respect to π is $\mathbb{C}1$ and that π does not extend to an isomorphism $\mathrm{L}(\Lambda) \to M$. If $N \subset M$ is a von Neumann algebra then*

$$\{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \mid g \in \Lambda\}' \cap \mathrm{L}^\infty(Y)\overline{\otimes}\mathcal{B}(\mathrm{L}^2(N)) = 1 \otimes N \, .$$

Proof of Theorem 6. Let $\mathcal{Q} = \{\sigma_\lambda \otimes (J\mathrm{E}_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}''$. For $\lambda \in \Lambda$ and $O \subset G$ a neighbourhood of the identity, we let $K_{\lambda,O}$ be the $\|\cdot\|_2$-convex closure of $\{\pi(h\lambda h^{-1}) \mid h \in \Lambda \cap O\}$. Writing $K_\lambda = \bigcap_O K_{\lambda,O}$, where the intersection runs over all open neighbourhoods of the identity, we obtain a non-empty $\|\cdot\|_2$-closed convex set in M.

Claim: For all $\lambda \in \Lambda$, we have $\tau(\pi(\lambda))1 \in K_\lambda$.
Denote by x_λ the unique $\|\cdot\|_2$-minimum of K_λ. If $(\lambda_n)_n$ is a sequence in Λ going to e in G, and O is a neighbourhood of e in G, then $\pi(\lambda_n)K_{\lambda,\lambda_n^{-1}O}\pi(\lambda_n^{-1}) = K_{\lambda,O}$ is well defined for big n. Hence for each neighbourhood O of the identity in G, there is $N \in \mathbb{N}$ such that for all $n \geq N$, the element $\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1})$ lies in $K_{\lambda,O}$. Since $\|\pi(\lambda_n)x_\lambda\pi(\lambda_n)\|_2 = \|x_\lambda\|_2$ for all $n \in \mathbb{N}$, any weak cluster point y of $(\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1}))_n$ satisfies $\|y\|_2 \leq \|x_\lambda\|_2$. By uniquenss of the $\|\cdot\|_2$-minimum in K_λ it follows that $y = x_\lambda$. We have shown that $\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1})$ converges weakly to x_λ. Using

$$\begin{aligned}\|\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1}) - x_\lambda\|_2 &= \|\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1})\|_2 + \|x_\lambda\|_2 + 2\mathrm{Re}\langle\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1}), x_\lambda\rangle \\ &= 2\|x_\lambda\|_2 + 2\mathrm{Re}\langle\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1}), x_\lambda\rangle \, ,\end{aligned}$$

we infer that $\pi(\lambda_n)x_\lambda\pi(\lambda_n^{-1}) \to x_\lambda$ in $\|\cdot\|_2$. This shows that x_λ lies in the G-algebra of M, which equals $\mathbb{C}1$. So $x_\lambda = \tau(x_\lambda)1$. Since $\tau(\pi(h\lambda h)) = \tau(\pi(\lambda))$ for all $h \in \Lambda$, we see that τ is constant on K_λ, implying that $\tau(\pi(\lambda)) = x_\lambda \in K_\lambda$.

Claim: $\sigma_\lambda \otimes 1 \in \mathcal{Q}$ for all $\lambda \in \Lambda$ satisfying $\tau(\pi(\lambda)) \neq 0$.
Take $\lambda \in \Lambda$ such that $\tau(\pi(\lambda)) \neq 0$. Applying $\mathrm{id}\otimes\mathrm{Ad}(J)$, it suffice to show that $\sigma_\lambda \otimes 1$ is in the strong closure of $\mathrm{span}\{\sigma_\lambda \otimes \mathrm{E}_N(\pi(\lambda)) \mid \lambda \in \Lambda\}$. We will even show that $\sigma_\lambda \otimes 1$ is in the strong closure of the uniformly bounded set $\mathrm{conv}\{\sigma_\lambda \otimes \mathrm{E}_N(\pi(\lambda)) \mid \lambda \in \Lambda\}$. Take $\varepsilon > 0$ and finite families $F_1 \subset \mathrm{L}^2(Y)$ and $F_2 \subset \mathrm{L}^2(N)$. Making ε smaller if necessary, we may assume that all $\xi_1 \in F_1$ satisfy $\|\xi_1\|_2 \leq 1$. Moreover, since $\sigma_\lambda \otimes 1$ will be approximated by uniformly bounded elements, we can assume that $F_2 \subset N$ and $\|\xi_2\|_\infty \leq 1$ for all $\xi_2 \in F_2$. Since G acts continuously on Y there is an open neighbourhood O of the identity in G such that for all $g \in O$ and all $\xi_1 \in F$ the following estimate holds.

$$\|(\sigma_\lambda - \sigma_{g\lambda g^{-1}})\xi_1\|_2 < \varepsilon/2 \, .$$

By the previous claim, there is a convex combination $\sum_{i=1}^n c_i\pi(h_i\lambda h_i^{-1})$ such that $h_i \in O$ for all $i \in \{1, \dots, n\}$ and

$$\|\tau(\pi(\lambda)) - \sum_{i=1}^n c_i\pi(h_i\lambda h_i^{-1})\|_2 < \varepsilon/2 \, .$$

We obtain for all $\xi_1 \in F_1$, $\xi_2 \in F_2$ the estimate

$$
\begin{aligned}
&\|\big((\sigma_\lambda \otimes \tau(\pi(\lambda))) - \sum_{i=1}^n c_i \sigma_{h_i \lambda h_i^{-1}} \otimes \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\big)(\xi_1 \otimes \xi_2)\|_2 \\
&\leq \|\big((\sigma_\lambda \otimes \tau(\pi(\lambda))) - \sum_{i=1}^n c_i \sigma_\lambda \otimes \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\big)(\xi_1 \otimes \xi_2)\|_2 \\
&\qquad + \|\big(\sum_{i=1}^n c_i \sigma_\lambda \otimes \mathrm{E}_N(\pi(h_i \lambda h_i^{-1})) - \sum_{i=1}^n c_i \sigma_{h_i \lambda h_i^{-1}} \otimes \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\big)(\xi_1 \otimes \xi_2)\|_2 \\
&= \|\big(\sigma_\lambda \otimes \big(\tau(\pi(\lambda)) - \sum_{i=1}^n c_i \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\big)(\xi_1 \otimes \xi_2)\|_2 \\
&\qquad + \|\sum_{i=1}^n c_i \big((\sigma_\lambda - \sigma_{h_i \lambda h_i^{-1}}) \otimes \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\big)(\xi_1 \otimes \xi_2)\|_2 \\
&\leq \|\sigma_\lambda \xi_1\|_2 \|\tau(\pi(\lambda)) - \sum_{i=1}^n c_i \mathrm{E}_N(\pi(h_i \lambda h_i^{-1}))\|_2 \|\xi_2\| \\
&\qquad + \sum_{i=1}^n c_i \|(\sigma_\lambda - \sigma_{h_i \lambda h_i^{-1}}) \xi_1\|_2 \|\mathrm{E}_N(\pi(h_i \lambda h_i^{-1})) \xi_2\|_2 \\
&\leq \|\tau(\pi(\lambda)) - \sum_{i=1}^n c_i \pi(h_i \lambda h_i^{-1})\|_2 + \sum_{i=1}^n c_i \|(\sigma_\lambda - \sigma_{h_i \lambda h_i^{-1}}) \xi_1\|_2 \\
&\leq \varepsilon/2 + \sum_{i=1}^n c_i \varepsilon/2 \\
&= \varepsilon \,.
\end{aligned}
$$

Since simple tensors span $\mathrm{L}^2(Y) \otimes \mathrm{L}^2(N)$, we see that $\tau(\pi(\lambda))(\sigma_\lambda \otimes 1) = (\sigma_\lambda \otimes \tau(\pi(\lambda))) \in \mathcal{Q}$. Now $\tau(\pi(\lambda)) \neq 0$, implies that $(\sigma_\lambda \otimes 1) \in \mathcal{Q}$.

Since $\{\lambda \in \Lambda \mid \tau(\pi(\lambda)) \neq 0\}$ is conjugation invariant, it generates a normal subgroup N of G. By assumption on τ, the group N is non-trivial. So the simplicity assumption on G shows that N is dense. So there is a subgroup $\Lambda_0 \leq \Lambda$ that is dense in G and such that $\sigma_\lambda \otimes 1 \in \mathcal{Q}$ for all $\lambda \in \Lambda_0$. Hence, continuity of $G \curvearrowright \mathrm{L}^\infty(Y)$ implies $\sigma_g \otimes 1 \in \mathcal{Q}$ for all $g \in G$. In particular, $\sigma_\lambda \otimes 1 \in \mathcal{Q}$ for all $\lambda \in \Lambda$. So also $1 \otimes J\mathrm{E}_N(\pi(\lambda))J \in \mathcal{Q}$ for all $\lambda \in \Lambda$.

Ergodicity of $G \curvearrowright \mathrm{L}^\infty(Y)$ implies $\mathcal{Q}' \cap \mathrm{L}^\infty(Y) \overline{\otimes} \mathcal{B}(\mathrm{L}^2(N)) \subset 1 \otimes \mathcal{B}(\mathrm{L}^2(N))$. Moreover $\{\mathrm{E}_N(\pi(\lambda)) \mid \lambda\}'' = \mathrm{E}_N(M) = N$. So we infer that

$$\mathcal{Q}' \cap (1 \otimes \mathcal{B}(\mathrm{L}^2(N))) = (1 \otimes \mathcal{B}(\mathrm{L}^2(N))) \cap (\mathrm{L}^\infty(B) \overline{\otimes} N) = 1 \otimes N$$

This finishes the proof. □

6 Outer actions

In this section we explain outer automorphisms of von Neumann algebras. This notion is central in the proof of Theorem 5, which is presented in Section 7.

Definition 10 (Inner automorphisms)**.** Let M be a von Neumann algebra and $\alpha \in \mathrm{Aut}(M)$. Then α is called inner if there is a unitary $u \in \mathcal{U}(M)$ such that $\alpha(x) = (\mathrm{Ad}\, u)(x) = uxu^*$.

Remark 11. If M is a factor and α is an inner automorphism, then $\alpha = \operatorname{Ad} u$ for a unique unitary up to multiplication by scalars. Indeed, if $\operatorname{Ad} u = \operatorname{Ad} v$ for some unitaries $u, v \in M$, then $\operatorname{Ad}(v^*u) = \mathrm{id}$ implies $v^*u \in \mathcal{Z}(M) = \mathbb{C}1$.

Definition 12 (Outer action)**.** Let M be a von Neumann algebra and $\alpha \in \operatorname{Aut}(M)$. Then α is outer if whenever there is $x \in M$ such that for all $y \in M$ the formula $xy = \alpha(y)x$ holds, then $y = 0$.

An action $G \overset{\alpha}{\curvearrowright} M$ of a discrete group on a von Neumann algebra is outer if every α_g, $g \in G \setminus \{e\}$ is outer.

One can interpret this definition by saying that an outer action is as far as possible from being implemented by inner automorphisms. The following proposition makes this point of view clear.

Proposition 13. *Let $\alpha \in \operatorname{Aut}(M)$ be an automorhpism of a von Neumann algebra. Then there is a central projection $p \in \mathcal{Z}(M)$ such that*

- $\alpha(p) = p$*,*
- $\alpha|_{pM}$ *is inner and*
- $\alpha|_{(1-p)M}$ *is outer.*

In particular, every automorphism of a factor is either inner or outer.

Proof. Let $p \in \mathcal{Z}(M)$ be the maximal α-invariant projection such that $\alpha|_{pM}$ is inner. We have to show that $\alpha|_{(1-p)M}$ is outer. Replacing M by $(1-p)M$ it hence suffices to show the following statement.

Claim. Whenever α is an automorphism of M that is not outer, then there is a central α-invariant projection $p \in \mathcal{Z}(M)$ such that $\alpha|_{pM}$ is inner.
Since α is not outer, there is a non-zero element $a \in M$ such that $ax = \alpha(x)a$ for all $x \in M$. Since

$$a^*ax = a^*\alpha(x)a = xa^*a\,,$$

we see that $a^*a \in \mathcal{Z}(M)$. If $a = v|a|$ denotes the polar decomposition of a, then for all unitaries $u \in \mathcal{U}(M)$ we have

$$a = \alpha(u)^*au = \alpha(u)^*v|a|u = \alpha(u)^*vu|a|\,.$$

Since $a^*a \in \mathcal{Z}(M)$, also $v^*v = \operatorname{supp}(a^*a) \in \mathcal{Z}(M)$. This implies $(\alpha(u)^*vu)^*\alpha(u)^*vu = u^*v^*vu = v^*v$. By uniqueness of the polar decomposition of a, it follows that $\alpha(u)^*vu = v$. Put differently, we have $vu = \alpha(u)v$ for all $u \in \mathcal{U}(M)$. Hence $vx = \alpha(x)v$ for all $x \in M$.

We show that $vv^* = v^*v$. Since v^*v is central, we have $vv^* \leq v^*v$. Now note that $\alpha(v^*v) \in \mathcal{Z}(M)$ satisfies

$$v^*v\alpha(v^*v) = v^*\alpha(v^*v)v = v^*v(v^*v) = v^*v\,.$$

So $v^*v \leq \alpha(v^*v)$. Moreover, since v^*v is central, we have

$$v\alpha^{-1}(v^*v - vv^*) = (v^*v - vv^*)v = v - v = 0\,,$$

implying that $\alpha(v^*v)(v^*v - vv^*) = 0$. Since $vv^* \leq v^*v \leq \alpha(v^*v)$, it follows that

$$v^*v - vv^* = \alpha(v^*v)(v^*v - vv^*) = 0\,.$$

This shows that $vv^* = v^*v \in \mathcal{Z}(M)$.

As a byproduct of the last paragraph, we saw that $\alpha(v^*v) \geq v^*v$. It follows that $\alpha^n(v^*v) \geq \alpha^{n-1}(v^*v)$ for all $n \geq 1$. So $p = v^*v + \sum_{n\geq 1}(\alpha^n(v^*v) - \alpha^{n-1}(v^*v))$ is an α-invariant central projection in M. Moreover, the partial isometry

$$u = v + \sum_{n\geq 1} \alpha^n(v)(\alpha^n(v^*v) - \alpha^{n-1}(v^*v))$$

satisfies $u^*u = uu^* = p$. For $n \geq 1$ and $x \in M$, we obtain the equality

$$\alpha^n(v)x = \alpha^n(v\alpha^{-n}(x)) = \alpha^n(\alpha^{-(n-1)}(x)v) = \alpha(x)\alpha^n(v)\,,$$

showing that $ux = \alpha(x)u$ for all $x \in M$. We have shown that $\alpha|_{pM}$ is inner, completing the proof. $\square$

Remark 14. Outerness must be considered as a generalisation of freeness for non-singular actions. More precisely, if α is an automorphism of an abelian von Neumann algbra $A = \mathrm{L}^\infty(X)$, then α is outer if and only if the associated automorphism α_0 of X is free.

Indeed, assume that α is outer. Let $Y = \{x \in X \mid \alpha_0(x) = x\}$ and write $p = \mathbb{1}_Y$. Then $\alpha(q) = q$ for all $q \leq p$. So $xp = \alpha(x)p = p\alpha(x)$ for all $x \in A$. It follows that $p = 0$, or equivalently Y is a negligible set.

If α_0 is free, let p be a projection of A such that α is inner on pA. Since pA is abelian, α is trivial on pA. Take $Y \subset X$ such that $p = \mathbb{1}_Y$. For almost every $x \in Y$ we have $\alpha_0(x) = x$, so Y is negligible. It follows that $p = 0$.

7 G-algebras

In this section we prove Theorem 5. The treatment of outer actions in the previous section will allow us to split up its proof into the case of abelian von Neumann algebras and factors.

The key ingredient in the proof of Theorem 5 is the following proposition.

Proposition 15 (Proposition 4.1 in [CP13])**.** *Let G be a simple non-compact Lie group with trivial centre. Suppose that $\Lambda \leq G$ is a countable dense subgroup. If $\alpha : G \to \mathrm{Aut}(M,\tau)$ is a continuous trace preserving ergodic action of G on a non-trivial tracial von Neumann algebra M, then the restriction of α to Λ is outer.*

We will argue that in order to prove this proposition, it suffices to consider the two special cases where M is abelian and where M is a factor.

Proposition 16. *Let G be a simple connected non-compact Lie group with trivial centre. Suppose that $\Lambda \leq G$ is a countable dense subgroup. If $\alpha : G \to \mathrm{Aut}(X,\mu)$ is a continuous ergodic probability measure preserving action of G on a non-trivial probability measure space X, then the restriction of α to Λ is free.*

Proposition 17. *Let G be a simple connected non-compact Lie group with trivial centre. Suppose that $\Lambda \leq G$ is a countable dense subgroup. If $\alpha : G \to \mathrm{Aut}(M,\tau)$ is a continuous ergodic action of G on a finite factor M, then the restriction of α to Λ is outer.*

Proof that Propositions 16 and 17 imply Proposition 15. Let $G \overset{\alpha}{\curvearrowright} M$ be given as in the statement of Proposition 15. Assume that there is $g \in \Lambda$ such that α_g is not outer. By Proposition 13 there is a non-zero central α_g-invariant projection $p \in \mathcal{Z}(M)$ such that $\alpha_g|_{pM}$ is inner. Since, $\alpha_g|_{\mathcal{Z}(pM)}$ is trivial, it follows that $\alpha_g|_{\mathcal{Z}(M)}$ is a non-free action. So Proposition 16 shows that $\mathcal{Z}(M) = \mathbb{C}1$, meaning that M is a factor. We can hence apply Proposition 17 so as to obtain a contradiction. This finishes the proof. □

We will not give a proof in the abelian case (Proposition 16). It can be found in [CP12, Section 7]. We concentrate on the factor case instead.

Proof of Proposition 17. Let $G \overset{\alpha}{\curvearrowright} M$ be as in the statement of the proposition. Since M is a factor, Proposition 13 says that every α_g, $g \in G$ is either outer or inner. We assume for a contradiction that the group $I = \{g \in G \,|\, \alpha_g \text{ is inner}\}$ is non-trivial. For $g \in I$ let $u_g \in M$ be some unitary satisfying $\alpha_g = \operatorname{Ad} u_g$. If $h \in G$, then $\alpha_{hgh^{-1}} = \alpha_h \circ \operatorname{Ad} u_g \circ \alpha_h^{-1} = \operatorname{Ad}(\alpha_h(u_g))$. So I is normal in G. By topological simplicity, it follows that I is dense in G.

For $g \in I$ and $x \in M$, we have

$$\operatorname{Ad}(\alpha_g(u_g))(x) = \alpha_g(u_g) x \alpha_g(u_g)^* = \alpha_g(u_g \alpha_g^{-1}(x) u_g^*) = \alpha_g(x) \,.$$

By Remark 11, the element u_g is unique up to multiplication with elements in S^1. So it follows that $\alpha_g(u_g) \in \mathrm{S}^1 u_g$. Hence $\alpha_h(u_g) \in \mathrm{S}^1 u_g$ for all $h \in \overline{\langle g \rangle}$. By definition of ergodicity of $G \curvearrowright M$ we know that $G \overset{\alpha}{\curvearrowright} \mathrm{L}^2(M) \ominus \mathbb{C}1$ doesn't have a fixed vector. As G has the Howe-Moore property, we infer that $\overline{\langle g \rangle}$ must be compact. We showed that there is dense subset of G each of whose elements generates a compact subgroup. This contradicts [Pla68]. So we reached a contradiction with the assumption that $G \curvearrowright M$ is not outer. □

We can now prove that G-algebras in the setting of Theorem 3 are trivial.

Theorem 5 (Recall) **.** *Let G be a simple non-compact Lie group with trivial centre. Assume that $\pi : \Lambda \to \mathcal{U}(M)$ is a non-degenerate finite factor representation of Λ. Then the G-algebra of π equals $\mathbb{C}1$.*

Proof. Let $\pi : \Lambda \to \mathcal{U}(M)$ be a non-degenerate finite factor representation and denote by $M_0 \subset M$ the G-algebra of π. Since M is a factor, $G \curvearrowright M_0$ is ergodic. Denote by $\mathrm{E} : M \to M_0$ the trace preserving conditional expectation. Then for all $x \in M_0$ and all $\lambda \in \Lambda$, we have

$$\mathrm{E}(\pi(\lambda))x = \mathrm{E}(\pi(\lambda)x) = \mathrm{E}(\operatorname{Ad}(\pi(\lambda)(x)\pi(\lambda)) = \operatorname{Ad}(\pi(\lambda)(x)\mathrm{E}(\pi(\lambda)) \,.$$

By Proposition 15, the action of Λ on M_0 is outer, so $\mathrm{E}(\pi(g)) = 0$. Since $\operatorname{span} \pi(\Lambda)$ is weakly dense in M, it follows that $M_0 = 0$, which is absurd. □

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